Math 3236 Statistical Theory

1/26/23

Beta distribution and Bayes Updates

(sect 5.8, 7.1 & 7.2)

Coin flip: an experiment with two outcomes: 0, 1

Probability of 1, called p.

Assume that we know nothing at the beginning that is our prior distribution on p is uniform.

$$f(p) = \begin{cases} 1 & 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{prior}$$

Make one exp. and The resut is

$\mathcal{L}$:

$$f(p \mid \mathcal{L}) = \frac{\mathbb{P}(\mathcal{L} \mid p)\, f(p)}{\mathbb{P}(\mathcal{L})}$$

$$= 2p$$

posTerior

$$f(p \mid 0) = 2(1-p)$$

I flip my coin N Times

and I get n Heads ($\mathcal{L}$)

$$f_N(p \mid \underset{\text{number of } H}{\underline{n}}) = \frac{\cancel{\binom{N}{n}}\, p^n (1-p)^{N-n}}{\int \cancel{\binom{N}{n}}\, p^n (1-p)^{N-n}}$$

$$f(p \mid n) = \frac{p^n (1-p)^{N-n}}{\int_0^1 p^n (1-p)^{N-n}}$$

Beta distribution.

Def: Beta function:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$$

Th:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$B(n, m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$$

$$\int_0^1 x^{n-1} (1-x)^{m-1} dx = \frac{(n-1)!(m-1)!}{(n+m-1)!}$$

$$\int x^{n} (1-x)^{m} \, dx =$$

$$\frac{m}{n+1} \int x^{n+1} (1-x)^{m-1} \, dx \quad \ldots$$

$$x^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha} e^{-tx} \, dt$$

$$f(x \mid \alpha, \beta) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha, \beta)} =$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

If $X \sim \mathrm{Beta}(\alpha, \beta)$ Then

$$\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$$

$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$B(1,1) =$ uniform

If we start our experiment with the prior that

$$p \sim B(1,1)$$

$f(p,n)$ beta dist.

with $\alpha = n+1 \quad \beta = N-n+1$

Suppose now that my prior is

$$B(\alpha, \beta)$$

after observing n f in N attempts The posterior is $B(\alpha + n, \beta + N - n)$

Bernoulli and B are conjugate family

Updating rule for a Bern. exp. Takes a B dist and returns a B dist.

Since a B dist. is identified by the two par. α, β, the update can be written as an updating rule on α, β

$$\alpha \to \alpha + n$$

$$\beta \to \beta + N - n$$

$$\mathbb{E}(P) = \frac{1}{2} \qquad \frac{\alpha}{\alpha+\beta} = \frac{1}{2}$$

$$\alpha = \beta \implies \mathbb{E}(P) = \frac{1}{2}$$

The larger α The more convinced you are That The coin is fair.

$$\mathbb{E}(P \mid n) = \frac{\alpha + n}{\alpha + \beta + N}$$

if $\alpha = \beta$

$$= \frac{\alpha + n}{2\alpha + N}$$

if The coin is actualli fair

Then, in propability,

$n = \frac{N}{2}$ for N large.

$\mathbb{E}(P, n)$ converge in probability to $\frac{1}{2}$ when $N \to \infty$.

If prob of H is p

Then, in prob,

$$n = Np$$

$$\lim_{N \to \infty} \frac{\alpha + Np}{\alpha + \beta + N} = p$$

If I estimate q

$\min_q \mathbb{E}((P - q)^2 \mid n)$ is reached

for $q = \mathbb{E}(P \mid n)$.

Ex:

$$\min_m \mathbb{E}\left((X-m)^2\right) = \mathrm{Var}(X)$$

$$\arg\min = \mathbb{E}(X)$$

If cost $\mathbb{E}(|P-q|)$

Then the best q = median of $P \mid n$

$$B(1,1) \xrightarrow{0} B(2,1) \xrightarrow{1} B(2,2)$$

$$B(1,1) \xrightarrow[\text{1 one}]{\text{1 zero}} B(2,2)$$

X_i are a Bernoulli r.s.

$$\mathbb{E}(X_i) = p$$

$$\hat{p} = \frac{1}{N} \sum_{i=1}^{N} X_i = \bar{X}$$

$$\mathbb{E}(\hat{p}) = p$$

$$\hat{p} \rightsquigarrow p$$

for N large

$$\frac{\sqrt{N}}{\sqrt{p(1-p)}}\left(\bar{X} - p\right) \Longrightarrow N(0,1)$$

$$B(Np, N(1-p)) \underset{N\to\infty}{\Longrightarrow} ??$$